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Weakly nonlinear EHD stability of slightly viscous jet

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ABSTRACT

A weakly nonlinear approach is utilized here to study the electrohydrodynamic (EHD) instability of an incompressible viscous liquid jet stressed by an axial electric field. The linear motion equations is solved in the light of nonlinear boundary conditions. The viscosity is assumed to be small. The study takes into account both the shear and radial components of the stresses at the interface. In the linear theory, we discuss the breakup phenomena of liquid jets. Also, it is found that, the electrical shearing stresses have no effect at the linear marginal state, while the linear cutoff wavenumber depends on the electrical shearing stresses. A nonlinear perturbation method is introduced. This method can be described our problem precisely. The nonlinear stability is compared with the linear stability condition in the weak viscosity case. It is found that, the weak viscosity has effect on the nonlinear stability condition, in contrast with the linear analysis, whereas the nonlinear cutoff wavenumber doesn't depend on the weak viscosity in both the linear and nonlinear theory.

Contents

1. Introduction

Electrohydrodynamic (EHD) is the study of fluid motions driven by external electrostatic fields. The process of EHD is dependent on so many parameters and physical properties of the fluid system. There has been continued interest in the behavior of liquid jets under the influence of an electric, because of the numerous industrial applications, such as in paint spraying [1], electronic ink-jet printers [2], etc. The study of capillary liquid jet instability using hydrostatic theory, first explained by Plateau [3]. He showed that the axisymmetric deformation is stable or unstable according as the wavelength of deformation of the cylindrical surface is less than or greater than the circumference of the cylinder. Rayleigh [4] extended Plateau's work using hydrodynamic theory of linear stability. He developed the important concept of the mode of maximum instability by treating liquids as perfect conductors. Most studies have tended to consider either perfect conductors [3,4] or perfect dielectrics [5]. Taylor [6] proposed an EHD theory based on the leaky dielectric model. This model accounts for the charge accumulation at the interface due to finite conductivities in the fluids, where the surface tangential electric stresses induce fluid motion. In the light of linear theory of leaky dielectric theory Melcher and Taylor [7] explained certain paradoxical phenomena pertaining to nonconducting fluids. Saville [8] examined the linear EHD stability of an infinite fluid cylinder in the presence of an axial electric field. Both fluids were treated as leaky dielectrics. He showed that a leaky dielectric requires much lower field strength than a perfect dielectric for jet stabilization to take place. In addition he showed that the stability of the cylindrical configuration depends on the relative magnitude of the conductivity and dielectric constant ra-

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tios. Experiment have been reported by Sankaran and Saville [9] on leaky dielectric bridge. Their results showed that, the stability with the application of an electric field depending on the conductivities and dielectric constants of the two media as predicted by the leaky dielectric model [8]. Burcham and Saville [10] studied the stability of a leaky dielectric bridge penned between planar electrodes held at different potentials and surrounded by a non-conducting, dielectric gas. The stability conditions of the perturbed system are discussed both theoretically and numerically. Burcham and Saville [11] compared the theoretical results with experimental work, that demonstrated how an electric field stabilizes an otherwise unstable configuration. Pelekasis et al. [12] studied the linear oscillations of viscous, capillary bridge in the presence of an axial electric field. They obtained, the stability conditions for both cases of leaky and perfect dielectrics. López-Herrera et al. [13] studied the linear electric viscous jets. They discussed the role of limited conductivity and permittivity on the behavior of electrified jets for viscosity limit, low and high electrical conductivity and permittivity. Elcoot [14] investigated the effect of a uniform surface charge in the presence of a finite rate of charge relaxation of cylindrical interface. He examined the effects of the surface charge and charge relaxation on the stability of the flow by considering various limiting cases in axisymmetric and nonaxisymmetric modes. He predicted a new unstable regions.

The nonlinear problem of the leaky dielectric model has attracted the attention of many investigators. By treating nonlinear processes rigorously, Feng and Scott [15] improved agreement between the theory and experiment for higher field strengths and larger deformations. Feng [16] extended the computations of Feng and Scott [15] to include the charge convection effect that is expected to emerge when the flow intensity is considerable. Theoretical treatments of the nonlinear aspects of the effect of an axial electric field on the streaming instability of surface waves, which propagating through porous media of a cylindrical flow of two concentric finitely conducting fluids, have been investigated by Elcoot and Moatimid [17]. They showed that the nonlinear theory predicted more accurately the instability, where new instability regions, appeared due the nonlinear effects. The nonlinear electro-viscous potential flow analysis has been studied by Elcoot [18]. He showed that, the nonlinear stability condition depend on the viscosity coefficients, which does not explain in the linear theory of viscous potential flow analysis model of Funada and Joseph [19,20]. In their model, they considered the normal stress is not neglected, but the effect of shear stress is neglected. Elcoot [21] introduced new technique based on the perturbation theory. He derived a new condition on the material properties, involving weak electric relaxation times in both fluids. Such effects can only be understood by nonlinear analysis, as the linear analysis fails to predict them.

A generalization of the nonlinear instability for viscous flow is a very difficult problem. The difficulty arises as the nonlinear terms are considerable. Fing and Bear [22] restrict themselves to the case of weak viscous effects. This weakness is regarded such that viscous effects appear at the interface and gradually decrease to be neglected in the bulk [23–25]. Their treatment based on the viscous or viscoelastic contribution has been demonstrated through the normal stress boundary condition. While, the tangential stress is ignored. In this paper we employ the nonlinear analysis based on the perturbation technique [21] to describe the stability of jet in the small viscosity case, under the influence of an axial uniform electric field. The study takes into account the shear and normal stresses effects at the interface.

2. Governing equations

We are interested in examining the stability of an infinite incompressible cylindrical jet of radius R, under the influence of an axial uniform electric field E_0. In what follows, the subscripts 1 and 2 denotes variables associated with the fluids inside and outside the jet, respectively. Bulk properties of the liquid (density ρ, viscosity μ, dielectric constant ε_1 and electric conductivity σ_1) as well as interfacial properties (surface tension T) are uniform and constant under the isothermal analysis. In the most practical applications the surrounding material is a gas and, thus it is assumed that it has negligible density and viscosity, but uniform and finite dielectric constant ε_2 and electric conductivity σ_2. The gravitational acceleration is ignored. The motion ensues from rest and the flow field generated due to wave motion. To describe the fluid motion, we use the moving frame of reference with the jet at rest. If (r, Z, t_0) is the coordinate system for the traveling jet and (r, z, t) for the jet rest, the transformation connecting the two systems is given by $z = Z - u_0 t_0, t = Z/u_0$ where u_0 is the uniform speed of jet along the axis of the cylinder, as [5,14]. For convenience, the usual cylindrical coordinates (r, θ, z) is used. Only the axisymmetric case is considered in this study. The interface between the liquid and gas is assumed to be well defined and initially cylindrical. The Maxwell equations lead to an exponential decay of the bulk charge density as $\exp(-\sigma_1 t/\varepsilon_1)$ where the parameter σ_1/ε_1 is sufficiently short, so that the electric charge density in the bulk is essentially zero. Therefore, the bulk forces of electrical origin are negligible and the field coupling occurs at the interface as specified by the appropriate boundary conditions [7].

The idea for the weakly nonlinear description is the some slight departure from the linear viewpoint [24,25]. At this end, the nonlinear problem will contain the linear description with some additional terms representing a correction for the main solution. The weakly nonlinear description given here depends on neglecting the nonlinear terms from equations of motion and applying the appropriate boundary conditions without dropping the nonlinear terms. At this stage, the dispersion relation should be extended to include nonlinear terms.

The electric potential Φ governed by Laplace equations

$$\nabla^2 \Phi_1 = 0, \tag{2.1}$$

$$\nabla^2 \Phi_2 = 0, \tag{2.2}$$

where ∇^2 is axisymmetric cylindrical Laplacian operator defined as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.3}$$

The Laplace equations (2.1) and (2.2) satisfied the requirement of steady-static charge conservation in the bulk fluid, as expressed in terms of zero divergence of electric current density due to Ohmic conduction.

The conservation equations of mass and momentum for the liquid jet are

$$\nabla \cdot \mathbf{v} = 0, \tag{2.4}$$

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla\Pi + \mu\nabla^2\mathbf{v}, \tag{2.5}$$

where $\mathbf{v}$ is the fluid velocity vector and Π is the modified pressure defined by

$$\Pi = p - \frac{1}{2}\varepsilon_j E_0^2, \quad j = 1, 2, \tag{2.6}$$

and p is the hydrostatic pressure. The relaxation time is much smaller than the liquid oscillation time. This prevents any free charge from appearing in the bulk of the liquid, and thus no electric stresses arise in Eq. (2.5).

We consider a perturbation, so that the surface deflection $S(r, z, t)$ is expressed by

$$S(r, z, t) = r - R - \eta = 0, \tag{2.7}$$

where

$$\eta=\delta\left(Ae^{i(kz-\omega t)}+\bar{A}e^{-i(kz-\omega t)}\right), \tag{2.8}$$

here A is an arbitrary parameter, which determines the behavior of the amplitude of the disturbance, with $\delta \ll 1$, $t>0$ is the time. $\eta=\eta(z,t)$ is the perturbation in the radius of the interface from its equilibrium value R, $i(=\sqrt{-1})$ is the imaginary number and $\bar{A}$ is the complex conjugate of the preceding terms. ω is being the (complex) frequency of the disturbance. The wavenumber k is real and positive. The unit outward normal vector $\mathbf{n}$ of the liquid jet can written in terms of the surface S takes the form

$$\mathbf{n}=\frac{\nabla S}{|\nabla S|}. \tag{2.9}$$

Therefore, the unit outward normal and tangential vectors $\mathbf{n}$ and $\mathbf{t}$, respectively in terms η are

$$\mathbf{n}=(1,0-ik\eta)\left[1-k^2\eta^2\right]^{-\frac{1}{2}}, \tag{2.10}$$

$$\mathbf{t}=(ik\eta,0,1)\left[1-k^2\eta^2\right]^{-\frac{1}{2}}. \tag{2.11}$$

For axisymmetric solution which is automatically satisfied by Eq. (2.4). We may define a stream function Ψ as

$$\mathbf{v}=(u_r,u_z)=-\frac{1}{r}\frac{\partial\Psi}{\partial z}e_r+\frac{1}{r}\frac{\partial\Psi}{\partial r}e_z, \tag{2.12}$$

where e_r and e_z are the unit vectors in the r and z-direction, respectively. Then, upon taking the curl of the linear momentum equation we have

$$\nabla^2\nabla^2\Psi=\lambda^2\nabla^2\Psi, \tag{2.13}$$

where $\lambda^2=-i\rho\omega/\mu$, we assume $k\ll\lambda$ and $|\lambda R|\gg 1$, the parameter λR can be viewed as in some sense as a Reynolds number. Under this condition, the vorticity equation (2.13) can be simplified when relevant properties of modified Bessel functions are used [26], and the stream function Ψ takes the form [27]:

$$\Psi=\left(A_1 rI_1(kr)+A_2Re^{\lambda(r-R)}\right)e^{i(kz-\omega t)}+O(\lambda R)^{-1}, \tag{2.14}$$

where A_1 and A_2 are arbitrary constants to be determined by appropriate nonlinear boundary conditions, and $I_m(kr)$ is a modified Bessel function of the first kind of order m. We have

$$\nabla\wedge\mathbf{v}=\varpi e_\phi, \tag{2.15}$$

where

$$\varpi=|\nabla\wedge\mathbf{v}|=-A_2\lambda^2Re^{\lambda(r-R)}+O(\lambda R)^{-1}. \tag{2.16}$$

The solution of the momentum Eq. (2.13) for the small viscosity case is given by Eq. (2.14). The first term in Eq. (2.14) corresponds to irrotational flow, while the second term represents the viscous effect. It is worth mentioning to note that, the viscous effects appear at the interface $r=R$. But, in the bulk $r<R$ the viscous effects gradually disappear as exponential decay of $e^{-\lambda|r-R|}$. Therefore, the viscous terms and vorticity perturbation ϖ are available at the boundary, but neglected in the bulk. This means that, the pressure is constant across the surface layer, and so the pressure is along the z-component of the inviscid momentum equation [19,20,22,27]. At this end, a weakly nonlinear idea [24,25] based on the boundary value problem has been solved in the light of nonlinear boundary conditions for linearized equations of motion.

The linear Laplace equations (2.1) and (2.2) are readily solved yielding

$$\Phi_1=B_1I_0(kr)e^{i(kz-\omega t)}, \tag{2.17}$$

$$\Phi_2=B_2K_0(kr)e^{i(kz-\omega t)}, \tag{2.18}$$

for some arbitrary constants B_1 and B_2 to be determined by the following nonlinear boundary conditions, and $K_m(kr)$ is a modified Bessel function of the second kind of order m.

At the free interface, $r=R+\eta(z,t)$, the solutions for Ψ and Φ_j have to satisfy the boundary conditions. The boundary conditions arising from physical arguments are as follows:

(i) Because the interface deforms, the kinematic boundary condition (it is required that the normal velocity of liquid must equal the velocity of the interface) hold and so

$$\frac{\partial\eta}{\partial t}=\mathbf{n}\cdot\mathbf{v}. \tag{2.19}$$

(ii) The jump in the tangential components of the electric field is zero across the interface, which leads to

$$\mathbf{t}\cdot\big|[\mathbf{E}]\big|=0, \tag{2.20}$$

where notation $|[\ ]|$ means the jump of a quantity across the interface, also in the outward direction, is defined as $|[f]|=f_2-f_1$.

(iii) Conservation of current crossing the liquid surface [10]:

$$\mathbf{n}\cdot\big|[\sigma\mathbf{E}]\big|=0. \tag{2.21}$$

(iv) Continuity of the axially directed shearing stress is

$$E_t\big|[\varepsilon E_n]\big|+\mu\left\{\mathbf{t}\cdot\left[(\mathbf{n}\cdot\nabla)\mathbf{v}\right]+\mathbf{n}\cdot\left[(\mathbf{t}\cdot\nabla)\mathbf{v}\right]\right\}=0, \tag{2.22}$$

where $E_n(=\mathbf{n}\cdot\mathbf{E})$ and $E_t(=\mathbf{t}\cdot\mathbf{E})$ represent the normal and the tangential components of the electric field, respectively. The electric field in both the liquid and gas is

$$\mathbf{E}_j=E_0\underline{e}_z-\nabla\Phi_j,\quad j=1,2. \tag{2.23}$$

(v) The dynamical condition that the normal stress should be continuous across the perturbed interface is

$$i\frac{\rho}{k}\frac{\partial u_z}{\partial t}-\rho\mathbf{v}^2+\frac{1}{2}\big|\left[\varepsilon E_n^2\right]\big|-\frac{1}{2}\big|\left[\varepsilon E_t^2\right]\big|+2\mu\mathbf{n}\cdot\left[(\mathbf{n}\cdot\nabla)\mathbf{v}\right] = T\nabla\cdot\mathbf{n}, \tag{2.24}$$

where, $\mathbf{v}^2=u_r^2+u_z^2$. The stress conditions (2.22) and (2.24) provide the coupling mechanism between the fluid flow quantities and the electrical quantities.

Substituting Eqs. (2.14), (2.17) and (2.17) into the nonlinear boundary conditions (2.19)–(2.22), we obtain the nonlinear constants A_1, A_2, B_1 and B_2 in terms of the elevation parameter η. Inserting these constants into the normal stress (2.24), one finds the following nonlinear characteristic function

$$D\left(\omega,k,\mu,|A|^2\right)=D(\omega,k,\mu)-\delta^2G|A|^2=0. \tag{2.25}$$

The linear characteristic function $D(\omega,k,\mu)$ represents the linear interaction parameters of the system is given later. The nonlinear term $G|A|^2$ in Eq. (2.25) arises by self- interaction and the coefficient G is the nonlinear term, the "Landau constant" is given in the Appendix A. It is the behavior of the physical parameters of the system in the nonlinear approach. The calculations are lengthy but straightforward, mostly of third order nonlinear. In order to facilitate the tedious computations, the MATHEMATICA package has been used.

To discuss the nonlinear stability problem for nonlinear characteristic function (2.25), we may introduced an modulation [21] to the problem, so that the linear dispersion relation $D(\omega,k,\mu)$ represents a slowly modulated wavetrain. To do this, first, we expand $D(\omega,k,\mu,|A|^2)$ as a function of ω, k, and $|A|^2$ in a Taylor series about the wavenumber k_0, the angular frequency ω_0, and the constant amplitude A_0, we obtain a series in power $\Omega=\omega-\omega_0$, $K=k-k_0$, where Ω and K are small. Therefore

$$D+\frac{\partial D}{\partial\omega}\Omega+\frac{\partial D}{\partial k}K+\frac{1}{2}\left[\frac{\partial^2 D}{\partial\omega^2}\Omega^2+2\frac{\partial^2 D}{\partial\omega\partial k}K\Omega+\frac{\partial^2 D}{\partial k^2}K^2\right] =\delta^2G|A|^2. \tag{2.26}$$

The nonlinear coefficient G evaluated at $A = A_0 = 0$, is given by $G = (\partial D/\partial|A|^2)|_{A_0}$. Suppose that

$$\Omega = \sum_{n=1}^{\infty} \delta^n \Omega_n, \qquad K = \sum_{n=1}^{\infty} \delta^n K_n, \tag{2.27}$$

where

$$\Omega_n = \Omega/\delta^n, \qquad K_n = K/\delta^n. \tag{2.28}$$

We can be rewritten η in the form

$$\eta = \delta A(z,t) e^{i(k_0 z - \omega_0 t)} + c.c. \tag{2.29}$$

Now, using the Fourier transform of the envelope function $A(\Omega, K)$ and its inverse transform $A(z,t)$ of the forms

$$A(\Omega, K) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(z,t) e^{i(\Omega t - Kz)}\, dt\, dz, \tag{2.30}$$

$$A(z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\Omega, K) e^{-i(\Omega t - Kz)}\, d\Omega\, dK, \tag{2.31}$$

with Eqs. (2.26)–(2.28) and (2.31) and equating coefficients of like powers of δ, we get the following set for D:

At the static (equilibrium) state is

$$p_0 = \frac{T}{R} - \frac{\varepsilon_2 E_0^2}{2}(\varepsilon - 1),$$

where p_0 is the equilibrium fluid hydrostatic pressure. It is clear from this equation at the fluid boundary $r = R$ surface, must satisfied the restriction $\varepsilon > 1$, at the limiting case $\varepsilon = 1$, the equilibrium fluid hydrostatic pressure p_0 must balance with interfacial tension. This means that, before perturbation, there is not allowed any free surface charge and it is seems that has been assumed that the fluid behaves as a perfect insulator.

At the first order we obtain

$$\begin{aligned} D(\omega, k, \mu) = {} & \frac{T}{R^2}(k^2R^2 - 1) + \frac{2\mu k^2(1 - kRI_a) + ikR\rho\omega I_a}{k^2 R F_1(2\mu k^2 - i\rho\omega)} \\ & \times \big[\rho\omega^2 F_1 - k^2\varepsilon_2 E_0^2\big(F_1(\varepsilon - 1) + (\sigma - 1) \\ & \times (\varepsilon K_a + I_a)\big)\big] + k\varepsilon_2 I_a K_a E_0^2 \frac{(\varepsilon - 1)(\sigma - 1)}{F_1}, \end{aligned} \tag{2.32}$$

where

$$I_a = I_0(kR)/I_0'(kR), \qquad K_a = -K_0(kR)/K_0'(kR), \tag{2.33}$$

$$F_1 = \sigma K_a + I_a, \qquad \varepsilon = \varepsilon_1/\varepsilon_2, \qquad \sigma = \sigma_1/\sigma_2. \tag{2.34}$$

In Eq. (2.33), the prime on the modified Bessel functions denotes differentiation with respect to r at $r = R$. It is clear that, the solution of the leaky dielectric problem (2.32) can be converted to that a perfect dielectric liquid in a perfect dielectric gas by setting $\sigma = \varepsilon$ and $\mu = 0$ [5]. In the leaky dielectric model, electromechanical coupling occurs only at the interface, where charge, carried to the interface by conduction, produces electric stresses different from those present in perfect dielectric. With dielectric stress is perpendicular to the interface, and curves the interface, this stress balance with interfacial tension. While, the leaky dielectrics are different because free charge accumulated on the interface modifies the field. The free charge density determines the tangential electric stress. This stress can only be balanced the viscous stress, is stated by Taylor and Melcher [7], and it is well known by the name "the leaky dielectric model of Taylor and Melcher" at Saville [28].

At the second and third order problems, we obtained the following system of partial differential equations

$$-\frac{\partial D}{\partial\omega}\frac{\partial A}{\partial t} + \frac{\partial D}{\partial k}\frac{\partial A}{\partial z} = 0, \tag{2.35}$$

$$\begin{aligned} & i\left(-\frac{\partial D}{\partial\omega}\frac{\partial A}{\partial t} + \frac{\partial D}{\partial k}\frac{\partial A}{\partial z}\right) + \frac{1}{2}\frac{\partial^2 D}{\partial\omega^2}\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 D}{\partial\omega\partial k}\frac{\partial^2 A}{\partial z\partial t} \\ & \quad + \frac{1}{2}\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 A}{\partial z^2} = \delta^2 G|A|^2 A, \end{aligned} \tag{2.36}$$

where the coefficients of the linear terms are simply derivatives of the linear characteristic function (2.32). Combining the two Eqs. (2.35) and (2.36) following similar arguments as given by Elcoot [21], we rewrite Eq. (2.36) to become

$$i\left(\frac{\partial A}{\partial\mathcal{T}} + \frac{d\omega}{dk}\frac{\partial A}{\partial\mathcal{Z}}\right) + \delta(P_r + iP_i)\frac{\partial^2 A}{\partial\mathcal{Z}^2} = \delta(Q_r + iQ_i)|A|^2 A, \tag{2.37}$$

where $\mathcal{Z} = z\delta$ and $\mathcal{T} = t\delta$ are slow space and time variables, and

$$P_r + iP_i = \frac{1}{2}\frac{d^2\omega}{dk^2}, \tag{2.38}$$

$$Q_r + iQ_i = \frac{G}{\partial D/\partial\omega}. \tag{2.39}$$

The nonlinear complex coefficients $(d^2\omega/dk^2)/2$ and $-G/(\partial D/\partial\omega)$ are the group velocity rate and the nonlinear interaction parameters, respectively. It is evident that, in the case of a coordinate system moving with the group velocity $d\omega/dk$, Eq. (2.37) needs to be rescaled. This can readily be accomplished by introducing the independent variables

$$\xi = \mathcal{Z} - (d\omega/dk)\mathcal{T} \quad \text{and} \quad \zeta = \mathcal{T}\delta, \tag{2.40}$$

so that in the $(\mathcal{Z}, \mathcal{T})$ coordinates equation (2.37) becomes

$$i\frac{\partial A}{\partial\zeta} + (P_r + iP_i)\frac{\partial^2 A}{\partial\xi^2} = (Q_r + iQ_i)|A|^2 A. \tag{2.41}$$

Eq. (2.37) may be used to study the stability behavior of the considered model. Lange and Newell [29] derived the stability criteria of this equation. If the solution of Eq. (2.37) is linearly perturbed, the perturbations are stable if both the following conditions

$$P_r Q_r + P_i Q_i > 0 \quad \text{and} \quad Q_i < 0, \tag{2.42}$$

where the real parts P_r and Q_r does not depend on the viscosity μ, while the imagery parts P_i and Q_i are function of μ.

3. Stability of a weakly viscous jet

It should be noted that Eq. (2.32) represents the linear dispersion relation for surface waves propagating through viscous liquid jet, as $D(\omega, k, \mu) = 0$. This dispersion relation is satisfied by the values of ω, k and μ and controls the stability in the linear problem. That is, each negative of the real part of ω corresponds to a stable mode of the interfacial disturbance. On the other hand, if the real part of ω is positive, the disturbance will grow in time and flow becomes unstable. For given values of the physical parameters of the system, most unstable wavenumber and corresponding growth rate may be readily found. We now proceed to examine asymptotic limit, when μ is a very small. The latter situation does not correspond to the complete neglect of viscous effects, for they are simply confined to a thin region near the interface. In this EHD boundary layer viscous effects balance the electrical shear stresses. Since the viscosity is a weak, it is mathematically assumed that $\mu \simeq \delta\hat{\mu}$ [22,27,30], which means that the time scale for viscous dissipation is much longer than the characteristic time of perturbed motion. In this limit, the linear characteristic function (2.32) yield

$$\begin{aligned} D_1 = {} & k\varepsilon_2 E_0^2 \frac{(\varepsilon - 1)(\sigma - 1)}{F_1} + \frac{kI_a\varepsilon_2 E_0^2}{F_1}\big[F_1(\varepsilon - 1) \\ & + (\varepsilon K_a + I_a)(\sigma - 1)\big] + \frac{T}{R^2}(k^2R^2 - 1) - \frac{\rho\omega^2 I_a}{k} = 0. \end{aligned} \tag{3.1}$$

We note that the normal stress with leaky dielectrics (3.1) differs from the perfect dielectrics, and there is also a tangential stress due to induced charge in Eq. (3.1). Before deformation the surface is free of charge since the field is parallel to the surface. Upon deformation, the normal stress is proportional to the first term of the linear characteristic function (3.1), and the tangential stress induced by the deformation is proportional to the second term of (3.1). So the sense of the stress depends on the relative magnitudes of both the electrical conductivities and the dielectric constants. However, the stability condition depends on the combined action of the two stresses. The gas surrounding the liquid jet can be considered as a leaky dielectric, when problem is associated with the existence of electrical effects on the gas surrounding the liquid jet. The solution of the leaky dielectric problem can lead to perfect dielectric liquid in a perfect dielectric gas, as $\sigma = \varepsilon$. In the special case of the neutral stability of the normal stress (when, $\sigma = \varepsilon$ and neglected the second term of the tangential stress of (3.1)) is mentioned earlier by Nayyar and Murty [5]. In this limit ($\mu \simeq \delta\hat{\mu}$), the solvability conditions (2.35) take the form

$$-\frac{\partial D_1}{\partial \omega}\frac{\partial A}{\partial t} + \frac{\partial D_1}{\partial k}\frac{\partial A}{\partial z} + i\mu\frac{\partial D_2}{\partial \mu}A = 0, \tag{3.2}$$

where

$$D_2 = -i\mu n_1,$$
$$n_1 = 2\frac{k}{\omega}(2I_a - \frac{1}{kR})\left[\omega^2 - \frac{k^2\varepsilon_2 E_0^2}{\rho F_1} \times \left\{F_1(\varepsilon - 1) + (\varepsilon K_a + I_a)(\sigma - 1)\right\}\right].$$

In inviscid case the term $\partial D_2/\partial\mu$ is vanish, and then Eq. (3.2) shows that the modulation propagates without change of the shape [31,32]. In the case of viscous effects, the temporal and the spatial rates are related by the group velocity $d\omega/dk$ and the term $\partial D_2/\partial\mu$. The solvability conditions (2.36) become

$$i\mu\left(-\frac{\partial^2 D_2}{\partial\mu\partial\omega}\frac{\partial A}{\partial t} + \frac{\partial^2 D_2}{\partial\mu\partial k}\frac{\partial A}{\partial z}\right) + i\left(-\frac{\partial D_1}{\partial\omega}\frac{\partial A}{\partial t} + \frac{\partial D_1}{\partial k}\frac{\partial A}{\partial z}\right) + \frac{1}{2}\frac{\partial^2 D_1}{\partial\omega^2}\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 D_1}{\partial\omega\partial k}\frac{\partial^2 A}{\partial z\partial t} + \frac{1}{2}\frac{\partial^2 D_1}{\partial k^2}\frac{\partial^2 A}{\partial z^2} - \frac{1}{2}\mu^2\frac{\partial^2 D_3}{\partial\mu^2}A = \delta^2 G_r|A|^2 A, \tag{3.3}$$

where

$$D_3 = \mu(2k^2/\rho\omega)^2 D_2, \quad \text{and} \quad G_r = G|_{\mu=0.}$$

Combining the two Eqs. (3.2) and (3.3), one obtains the following differential equation

$$i(1 - i\mu L)\frac{\partial A}{\partial t} + M_1\frac{\partial A}{\partial z} + P_r\frac{\partial^2 A}{\partial z^2} = Q_r|A|^2 A + \mu^2 RA, \tag{3.4}$$

the nonlinear coefficients P_r and Q_r are the real parts of P and Q, i.e. $P_r = P|_{\mu=0}$ and $Q_r = Q|_{\mu=0} = -G_r/(\partial D_1/\partial\omega)$, while the other coefficients are constructed in terms of the linear characteristic function relation as

$$L = i\frac{\partial^2 D_2}{\partial\mu\partial\omega}\bigg/\frac{\partial D_1}{\partial\omega},$$
$$M_1 = i\left[\frac{d\omega}{dk} - \frac{\partial^2 D_2}{\partial\mu\partial k} + \left(\frac{\partial^2 D_1}{\partial k\partial\omega} + \frac{\partial^2 D_1}{\partial\omega^2}\frac{d\omega}{dk}\right)\left(\frac{\partial D_2}{\partial\mu}\right)\left(\frac{\partial D_1}{\partial\omega}\right)^{-1}\right],$$
$$R = \frac{\partial D_3}{\partial\mu^2}\left(\frac{\partial D_1}{\partial\omega}\right)^{-1} - \frac{\partial^2 D_1}{\partial\omega^2}\left(\frac{\partial D_2}{\partial\mu}\right)^2\left(\frac{\partial D_1}{\partial\omega}\right)^{-3}.$$

We introduce the transformation [21] define as

$$\xi = \delta\left(z - M_1 t/(1 - i\mu L)\right), \qquad \zeta = \delta^2 t. \tag{3.5}$$

Under such transformations, Eq. (3.4) reduces to a nonlinear modified Schrödinger equation,

$$(i + \mu L)\frac{\partial A}{\partial\zeta} + P_r\frac{\partial^2 A}{\partial\xi^2} = Q_r|A|^2 A, \tag{3.6}$$

where the interaction term $\mu^2 RA$ eliminated by an appropriate transformation [33]. Eq. (3.6) is nonlinear modified Schrödinger equation, describes the nonlinear self- modulation of the capillary waves on a liquid jet, in the presence of an externally applied electric field, with a weak viscosity. In the special case, when $\mu \to 0$, Eq. (3.6) becomes standard nonlinear Schrödinger equation. From the known solutions of the standard nonlinear Schrödinger equation [34], it is interesting to note that, if the solution of linearly perturbed, the plane wave solution is unstable against modulation if $P_r Q_r < 0$. To investigate the stability conditions of Eq. (3.6), we follow the procedure of Elcoot [14] and [21], one obtains the nonlinear stability conditions

$$P_r Q_r > 0, \tag{3.7}$$

$$L > 0. \tag{3.8}$$

Condition (3.8) becomes

$$k^2\varepsilon_2 E_0^2\left[F_1(\varepsilon - 1) + (\varepsilon K_a + I_a)(\sigma - 1)\right] + \rho F_1\omega^2 > 0, \tag{3.9}$$

and with the aid of Eq. (3.1), we have

$$k\varepsilon_2 I_a K_a E_0^2\frac{(\varepsilon - 1)(\sigma - 1)}{F_1} + 2\frac{kI_a\varepsilon_2 E_0^2}{F_1}\left[F_1(\varepsilon - 1) + (\varepsilon K_a + I_a)(\sigma - 1)\right] + \frac{T}{R^2}(k^2R^2 - 1) > 0. \tag{3.10}$$

4. Stability near the critical curve

It is observed that the solution of Eq. (3.6) is not valid at $\partial D_1/\partial\omega = 0$. To determine a valid expansion in this limit, we express $\partial A/\partial z$ instead of $\partial A/\partial t$. Therefore, Eq. (3.2) yields

$$\frac{\partial A}{\partial z} + \frac{dk}{d\omega}\frac{\partial A}{\partial t} - i\mu\frac{\partial D_2}{\partial\mu}\bigg/\frac{\partial D_1}{\partial k}A = 0, \tag{4.1}$$

where $k' = dk/d\omega (= -(\partial D_1/\partial\omega)/(\partial D_1/\partial k))$ is the inverse of the group velocity. Eliminating the derivatives of z from (4.1) and (3.4) and rearrangement the result in terms of the original variables, one obtains

$$i\left(\frac{\partial A}{\partial z} + M\frac{\partial A}{\partial t}\right) - \frac{k''}{2(1 + \mu c_2)}\frac{\partial^2 A}{\partial t^2} = \frac{\delta^2}{(1 + \mu c_2)}\frac{G_r}{\partial D_1/\partial k}|A|^2 A, \tag{4.2}$$

where $k'' = d^2k/d\omega^2$ is the inverse of the group velocity rate and

$$M = \frac{k' + \mu c_1}{1 + \mu c_2},$$
$$c_1 = \frac{\partial D_2}{\partial\mu}\left(k'\frac{\partial^2 D_1}{\partial k^2} + \frac{\partial^2 D_1}{\partial k\partial\omega}\right) - \frac{\partial^2 D_2}{\partial\mu\partial\omega}\bigg/\frac{\partial D_1}{\partial k},$$
$$c_2 = \frac{\partial^2 D_2}{\partial\mu\partial k},$$
$$c_3 = (k')^2\frac{\partial^2 D_1}{\partial k^2}\bigg/\left(2\frac{\partial D_1}{\partial k}\right).$$

If we use the transformation

$$\mathsf{T} = \delta(t - zM), \qquad \mathsf{Z} = \delta^2 z, \tag{4.3}$$

then Eq. (4.2) takes the form

$$i(1 + \mu c_2)\frac{\partial A}{\partial\mathsf{Z}} - \frac{1}{2}k''\frac{\partial^2 A}{\partial\mathsf{T}^2} = \delta^2\frac{G}{\partial D_1/\partial k}A\bar{A}. \tag{4.4}$$

Eq. (4.4) is a nonlinear modified Schrödinger equation, which is valid near and in the critical state. In order to discuss the wave-train solutions of constant amplitude, we put

$$A = b\exp\bigl[i(KZ - \Omega T)\bigr], \tag{4.5}$$

where b is a constant. On substituting (4.5) into (4.4), we get the dispersion relation

$$\Omega^2 = -\frac{K + b^2\hat{G}}{\hat{P}}, \tag{4.6}$$

where $\hat{P} = -k''/2|_{k=k_c}$ and $\hat{G} = G/(\partial D/\partial k)|_{k=k_c}$. If $\hat{P}$ and $\hat{G}$ are both nonpositive and real, then for Ω to be imaginary we require $K < -b^2\hat{G}$. The nonlinear cutoff wavenumber is

$$K_n = k_c + \delta^2 b^2\hat{G} + O(\delta^4), \tag{4.7}$$

where k_c is the linear cutoff wavenumber, which is given by

$$k\varepsilon_2 I_a K_a E_0^2\frac{(\varepsilon-1)(\sigma-1)}{F_1} + \frac{kI_a\varepsilon_2E_0^2}{F_1}\bigl[F_1(\varepsilon-1) + (\varepsilon K_a + I_a)(\sigma-1)\bigr] + \frac{T}{R^2}(k^2R^2-1) = 0. \tag{4.8}$$

The linear cutoff wavenumber receives a second- order increment $b^2\hat{G}$. The effect of nonlinear is stabilizing if $b^2\hat{G}$ is negative, and vice versa. Eq. (4.7) shows that the nonlinear cutoff wavenumber does not depend on the weak viscosity.

5. Stability at the critical point

The transition from stability to instability takes place via a marginal state. In this section, we shall study the solvability conditions (3.2) and (3.3) at the critical point of the neutral curve (marginal state) in the parameter space, when the coefficients of Eq. (3.2) are vanish, i.e., $\partial D_1/\partial\omega = 0$, $\partial D_1/\partial k = 0$ and $\mu\partial D_2/\partial\mu = 0$. It is worthwhile to observe that the vanishing of the parameter $\mu\partial D_2/\partial\mu = 0$ is valid in the inviscid case ($\mu = 0$). Because we are interested in investigating the influence of electric field in the presence of weak viscosity, we intend to examine the marginal state when $\mu \neq 0$. The vanishing of $\partial D_2/\partial\mu = 0$ is available, whenever $n_1 = 0$ i.e.

$$\omega^2 = \frac{k^2\varepsilon_2E_0^2}{\rho F_1}\bigl[F_1(\varepsilon-1) + (\varepsilon K_a + I_a)(\sigma-1)\bigr]. \tag{5.1}$$

From Eqs. (3.1) and (5.1), we get the curve of neutral stability is

$$\frac{k\varepsilon_2E_0^2}{F_1}(\varepsilon-1)(\sigma-1) + \frac{T}{R^2}(k^2R^2-1) = 0. \tag{5.2}$$

This result is in agreement with the inviscid case, obtained by Elhefnawy et al. [35]. Also this result corresponds to the linear curve of the normal stress of neutral stability [28]. This means that, the marginal state in this case, occurs in the absence of a shear stress. Therefore, the solvability condition (3.2) in the neighborhood of the linear critical points become

$$\frac{1}{2}\frac{\partial^2 D_1}{\partial\omega^2}\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 D_1}{\partial\omega\partial k}\frac{\partial^2 A}{\partial z\partial t} + \frac{1}{2}\frac{\partial^2 D_1}{\partial k^2}\frac{\partial^2 A}{\partial z^2} = \delta^2\bigl(G_r|A|^2\bigr)A, \tag{5.3}$$

where G_r and the various derivative of D_1 are evaluated at the critical point.

Eq. (5.3) can be written in the form

$$\frac{\partial^2 A}{\partial t^2} - \mathcal{X}\frac{\partial^2 A}{\partial z^2} = \mathcal{J}A^2\bar{A}, \tag{5.4}$$

where

$$\mathcal{X} = \left(2\frac{d\omega}{dk}\frac{\partial^2 D_1}{\partial\omega\partial k} - \frac{\partial^2 D_1}{\partial k^2}\right)\Big/\frac{\partial^2 D_1}{\partial\omega^2}, \tag{5.5}$$

$$\mathcal{J} = (2G_r)\Big/\frac{\partial^2 D_1}{\partial\omega^2}. \tag{5.6}$$

Eq. (5.4) is the well-known nonlinear Klein–Gordon equation with real coefficients. In general the signs of both $\mathcal{X}$ and $\mathcal{J}$ play an important role to determine the stability condition, where the stability is achieved if both $\mathcal{X}$ and $\mathcal{J}$ are positive [36]. It is clear that $\mathcal{X}$ and $\mathcal{J}$ does not depend on the weak viscosity. This means that the effect of the viscous stress does not appear in the marginal state up to the third order of η. The nonlinear Klein–Gordon equation is encountered in other important applications such as the Kelvin–Helmholtz instability by Weissman [37] and the EHD streaming instability of cylindrical structures by Elhefnawy et al. [38]. The various cases of physical interest are studied by Murakami [39] and Elhefnawy [40]. Numerical solutions of Eq. (5.4) are presented, by Weissman [37], for the case of the nonlinear stabilization in one or two dimensions disturbance. Previous work by Weissman [37] and Murakami [39] is complemented and some corrections to the latter are pointed out by Parkes [41].

6. Numerical results

To support the analytical approach, we shall discuss in the next section, the numerical results in the light of linear and nonlinear approach, for a wide range of physical parameters of the problem. It is convenient for the linear and nonlinear numerical discussion to write the stability conditions in an appropriate dimensionless form. This can be done in a number of ways depending primarily on the choice of the characteristic length. Consider the following dimensionless forms: the characteristic length $= R$ (the radius of the undisturbed jet), the characteristic time $= [(\rho R^3)/T]^{\frac{1}{2}}$. The corresponding dimensionless quantities are given

$$k = k^*/R, \qquad \omega = \omega^*\left(\frac{T}{\rho R^3}\right)^{1/2},$$

$$E_0 = E_0^*\left(\frac{T}{\varepsilon_2 R}\right)^{1/2}, \qquad \mu = \mu^*/(\rho RT)^{1/2},$$

where superposed asterisks refer to dimensionless quantities. From now, it will be omitted for simplicity. The linear dispersion relation of (2.32) in the dimensionless form becomes:

$$(k^2-1) + \frac{2\mu k^2(1-kI_b) + ik\omega I_b}{k^2(\sigma K_b + I_b)(2\mu k^2 - i\omega)} \times \bigl[\omega^2(\sigma K_b + I_b) - k^2E_0^2\bigl((\varepsilon-1)(\sigma K_b + I_b) + (\sigma-1)(\varepsilon K_b + I_b)\bigr)\bigr] + kI_bK_bE_0^2\frac{(\varepsilon-1)(\sigma-1)}{\sigma K_b + I_b} = 0, \tag{6.1}$$

where $I_b = I_0(k)/I_0'(k)$ and $K_b = -K_0(k)/K_0'(k)$. In the absence of a viscosity and an electric field, the interface giving stability unless $k < 1$. The most unstable mode is $k \simeq 0.7$. These classical results were given by Rayleigh [4], demonstrating the instability of jets to long-wavelength axisymmetric disturbances which causes them to breakup into droplets. It is observed that the charged jets that occur in electrostatic spraying processes can exhibit greater stability than uncharged jets, through they too eventually breakup. The main purpose of Figs. 1–4 is to understand this phenomenon.

The linear dispersion relation (6.1) is a cubic equation in ω. We find that ω is generally complex, i.e., we can be written as $\omega = \omega_r + i\omega_i$, where ω_r and ω_i are real. This linear dispersion relation is transcendental, and can be solved for the growth rate ω_i as a function of k for physical parameters by a numerical approach. We did this by using MATHEMATICA, a computer code that can implement symbolic computation. The numerical results on

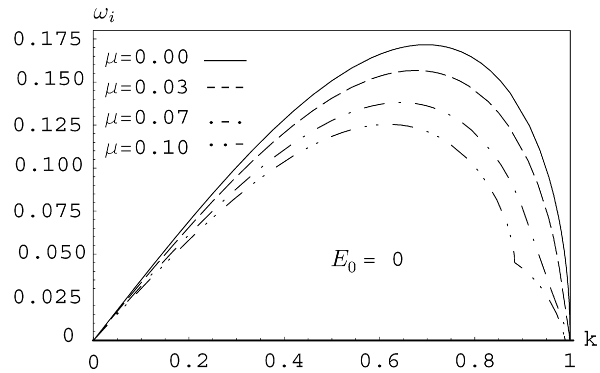

Fig. 1. Stability diagram in the $(\omega_i - k)$-plane, for linear system having $\varepsilon = 4$, $\sigma = 2$ and $E_0 = 0$. The graph indicates the relation (6.1) for different values of μ.

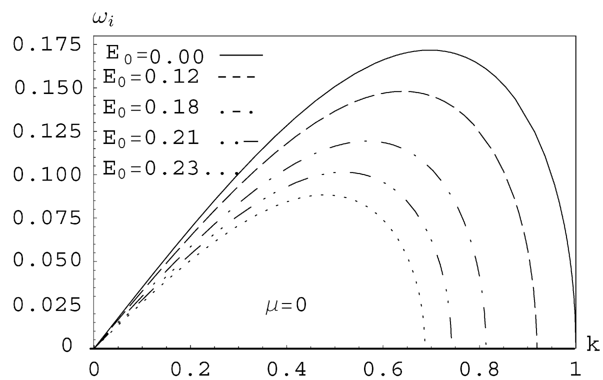

Fig. 2. Stability diagram for the same system as in Fig. 1, but with $\mu = 0$, for different values of E_0

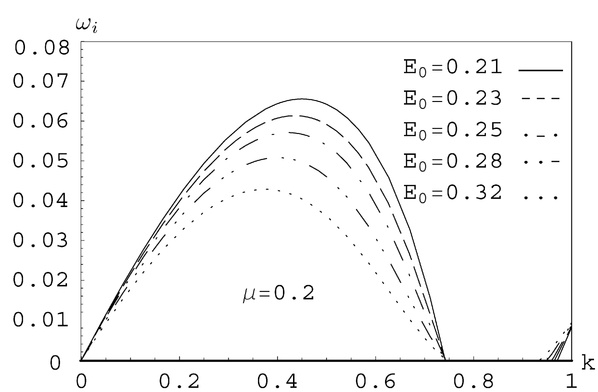

Fig. 3. Stability diagram for the same system as in Fig. 1, but with $\mu = 0.2$, for different values of E_0.

the variations of the growth rates for varying physical parameters are shown in Figs. 1–4. The numerical results inspect the maximum growth ω_i of the instability of the system.

In Fig. 1, we plot ω_i versus the wavenumber k for some values of μ when $E_0 = 0$ as shown in Fig. 1. It is found that, the viscosity reduces the magnitude of the growth rate for all wavenumbers. In addition, the maximum growth rate occurs at lower wavenumbers for more viscous jets. This is due to the more effective viscous damping at larger wavenumbers.

Fig. 2 represents the same system as considered in Fig. 1, but when $\mu = 0$ for some values of E_0. It is clear that, the liquid jet is unstable for axisymmetric disturbances with wavenumbers in the regime $k < 1$. Also we notice that, as E_0 increased the critical wavenumber k_c (at which the maximum growth rate occurs) de-

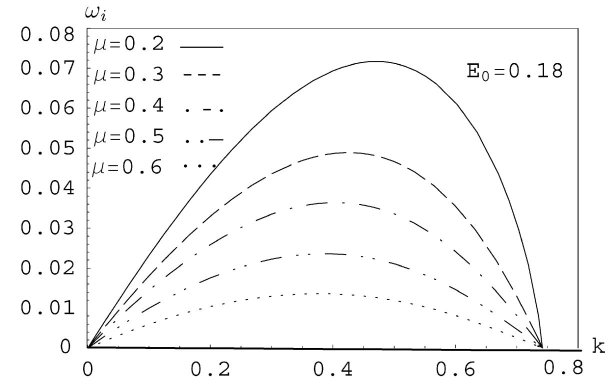

Fig. 4. Stability diagram for the same system as in Fig. 1, but with $E_0 = 0.2$, for different values of μ.

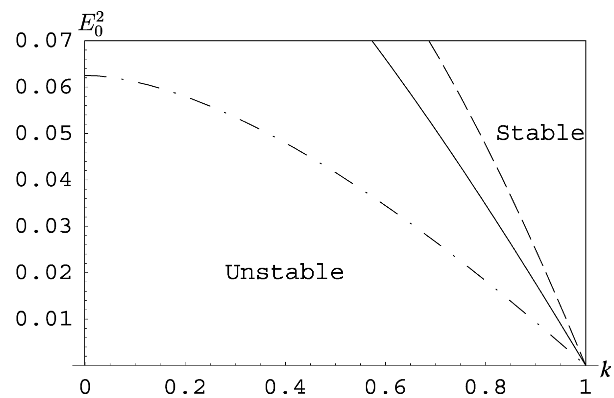

Fig. 5. Stability diagram in the $(E_0^2 - k)$-plane, for nonlinear system having $\varepsilon = 4$ and $\sigma = 2$. According to linear equations (6.2)–(6.4), nonlinear equations $P_r = 0$, $Q_r = 0$ and $D_1(2\omega, 2k) = 0$. The solid curve represents the linear curves (6.2), the dashed curve represents the curve (6.3) and the dashed dotted curve is the curve (6.4). The curves $P_r = 0$, $Q_r = 0$ and $D_1(2\omega, 2k) = 0$ lie in the negative part of E_0^2-axis.

creases. This means that, the electric field E_0 with $\mu = 0$, reduce the breakup phenomena of liquid jets into drops.

The effects which due to the coupling of electric field with viscosity are displayed in Figs. 3 and 4. It is found that, the interaction between the electric field and viscosity play an important role to describe the maximum growth rate of the instability of the system. In general, the critical wavenumber k_c decreases with increasing both E_0 and μ, because the linear neutral curve shifts downwards. Physically, the increase of the viscosity attempt prevention the jet to breakup points.

In the following figure, we shall discuss numerical analysis, the case of a weak viscosity for the linear and nonlinear solvability conditions (3.1) and (3.7), (3.8), respectively. Also, we compare the combined action of shear and normal stresses with the normal stress. The linear neutral stability relation (4.8) represents the combined action, while the relation (5.1) is the normal stress. In order to screen this examination, numerical calculations for the transition curves $P_r = 0$, $Q_r = 0$ as will as the second harmonic resonance curve $D_1(2\omega, 2k) = 0$, which arises because of the occurrence of zero divisors in the nonlinear interaction parameter Q_r, in addition, the transition curve $L = 0$. The transition curves $P_r = 0$, $Q_r = 0$ and $D_1(2\omega, 2k) = 0$, do not appear in Fig. 5 because it lies in the negative part of E_0^2-axis. The transition curves divide the E_0^2–k plane into stable and unstable regions as shown in Fig. 5. The linear neutral curve of the combined of shear and normal stresses corresponding to the linear characteristic function $D_1(\omega, k) = 0$ define by

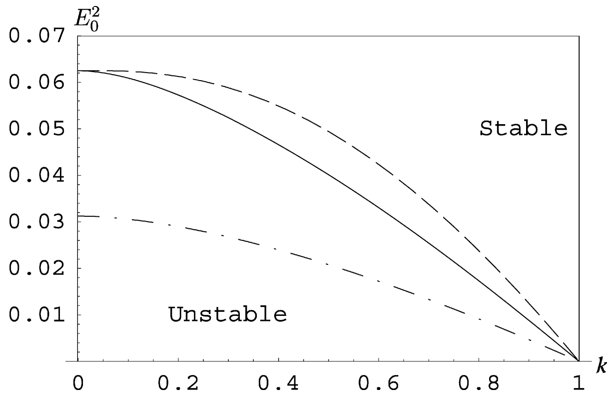

Fig. 6. Stability diagram for the same system as in Fig. 5, but with $\varepsilon = 8$.

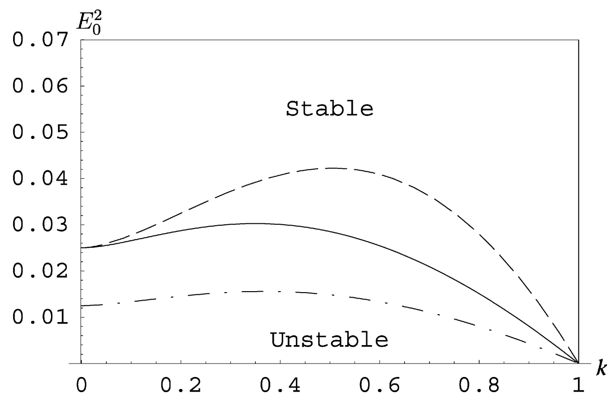

Fig. 7. Stability diagram for the same system as in Fig. 5, but with $\sigma = 8$.

$$E_{C1} = (k^2-1)(\sigma K_b + I_b)\big\{kI_b\big[(\varepsilon-1)(\sigma K_b + I_b) + (\sigma-1)(\varepsilon K_b + I_b) + K_b(\varepsilon-1)(\sigma-1)\big]\big\}^{-1}, \tag{6.2}$$

where E_{C1} represents, the critical electric field, separates stable regions from unstable regions. The interface of the system is linearly stable or unstable depend whether the electric field E_0^2 is larger or smaller than E_{C1}. The linear neutral curve of the normal stress corresponding Eq. (5.2) define by

$$E_{C2} = \frac{(k^2-1)(\sigma K_b + I_b)}{kI_b K_b(\varepsilon-1)(\sigma-1)}. \tag{6.3}$$

The solid line represents the linear curve (6.2). The dashed curve represents the linear curve (6.3). The dashed dotted line represents the nonlinear curve $L = 0$, which define by

$$E_{C3} = \frac{(\sigma K_b + I_b)(k^2-1)}{kI_b[(2(\varepsilon-1)(\sigma K_b + I_b) + (\sigma-1)(3K_b\varepsilon + (2I_b - K_b))]}. \tag{6.4}$$

Fig. 5 displays the stability diagram in the (E_0^2–k)-plane corresponding to the case $\varepsilon > 1$ and $\sigma > 1$, for $\varepsilon = 4$ and $\sigma = 2$. We observe that, the combined action of the two stresses make the jet greater stability than normal stress, with respect to the wavenumber. The nonlinear curve, made the stable region more expanded.

Fig. 6 represents the same system considered in Fig. 5, but when $\varepsilon = 8$. Comparing Fig. 5 and Fig. 6, we observe that the stable region increases, while the unstable region decreases, as the values of dielectric constant ratio $\varepsilon = \varepsilon_1/\varepsilon_2$increase, because the linear and nonlinear curves shift downwards. This means that, the electric field has a stabilizing effect with increasing ε. Fig. 7 represents the same system considered in Fig. 5, but when $\sigma = 8$. Comparing Figs. 5 and 6, we observe that, the electric field has a stabilizing effect with increasing $\sigma = \sigma_1/\sigma_2$.

We observe that, the combined action of the two stresses (slightly viscous) make the jet greater stability than normal stress (inviscid), with respect to the wavenumber. The nonlinear theory shows that, the electric field has strong stabilize effects, by adding a new nonlinear stable region to the linear stable region.

7. Conclusion

Weakly nonlinear scope is performed in order to investigate an incompressible viscous jet stressed by an axial uniform electric field. A weakly nonlinear idea based on the boundary value problem has been solved in the light of nonlinear boundary conditions for linearized equations of motion. The study takes into account the influence of the capillary force. The jet is surrounded by an incompressible gas that its hydrodynamic effect may be ignored. The viscosity is assumed to be small. The viscous and vorticity terms are neglected in the bulk (equation of motion), while at the boundary conditions are considered. Therefore, both the shear and normal stresses at the interface are considered. Both the liquid jet and gas have finite dielectric constants and electric conductivities. Hydrodynamic and electrostatic are coupled through the interfacial stresses, leading to a set of coupled partial differential equations. We obtained a dispersion relation for the first- order problem and a nonlinear Ginzburg–Landau equation, for the higher orders. These equations describing the stability analysis of the system. For a weak viscosity, we obtain two nonlinear modified Schrödinger equations describing the evolution of wave packets. One of these equations is used to analyze the stability of the system, while the other is used to determine the nonlinear cutoff electric field, separating stable and unstable regions. It is shown that the linear and the nonlinear cutoff wavenumber does not depend on the weak viscosity. Nonlinear Klein–Gordon equation describe a marginal state stability for a weak viscosity is obtained. It is found that at the marginal state, no electrical shearing stresses are present. Also, found that, the viscosity does not support the maximum of the growth rate of the instability of the system at the marginal state in both the linear and nonlinear theory.

Appendix A

The nonlinear coefficient of (2.25) is

$$\begin{aligned}
G = \frac{2}{D(2k, 2\omega, \mu)} \Bigg\{ \bigg(& \frac{1}{R^3} + \frac{k^2}{2R}\bigg) T + 2ik^2\mu(k\alpha_1 + s_1\alpha_2) \\
& - \varepsilon k^2 \varepsilon_2 E_0^2 (\sigma-1)\bigg[\frac{2\gamma_1}{I_a} + \frac{\gamma_2}{k} - \frac{\gamma_1^2}{2I_a^2}(\sigma-1) \\
& \times (1 + I_a^2)\bigg] + \varepsilon_2 k^2 E_0^2(\sigma-1)\bigg[\frac{2\delta_1}{I_a} + \frac{\delta_2}{k} \\
& - \frac{\delta_1^2}{2K_a^2}(\sigma-1)(1+K_a^2)\bigg] - \rho\omega\alpha_2 I_a + k\rho\omega\beta_1 \Bigg\}^2 \\
& + \Bigg\{ \bigg(\frac{3k^4}{2} - \frac{1}{R^4} - \frac{k^2}{2R^2}\bigg) T + 2i\mu k^2[s_1\alpha_3 + k\alpha_2 + k^2 s_1\alpha_1 \\
& + k^2\alpha_1 I_a] - \varepsilon k^2\varepsilon_2 E_0^2(\sigma-1)\bigg[\frac{\gamma_1\gamma_2}{I_a^2}(\sigma-1)(1+I_a^2) \\
& + \frac{2}{I_a}(\gamma_2 + \gamma_1^2) + \frac{\gamma_3}{k}\bigg] + k^2\varepsilon_2 E_0^2(\sigma-1)\bigg[\frac{\delta_1\delta_2}{K_a^2}(\sigma-1) \\
& \times \big(1 + K_a^2\big) + \frac{2}{K_a}(\delta_2 + \delta_1^2) + \frac{\delta_3}{k}\bigg] - \rho\omega\alpha_3 I_a + k\rho\omega\beta_2 \Bigg\},
\end{aligned}$$

where

$s_1 = I_a - 1/kR,$

$\gamma_1 = -I_a K_a / F_1,$

$$\gamma_2 = kI_a(F_1 - K_aF_3)/F_1^2,$$
$$\gamma_3 = k^2 I_a\big(F_1F_3 - K_a\big(F_1F_2 + F_3^2\big)\big)/F_1^3,$$
$$\delta_1 = -I_aK_a/F_1,$$
$$\delta_2 = kK_a(F_1 - I_aF_3)/F_1^2,$$
$$\delta_3 = -k^2K_a\big(F_1F_3 + I_a\big(F_1F_2 + F_3^2\big)\big)/F_1^3,$$
$$F_1 = \sigma K_a + I_a,$$
$$F_2 = \sigma I_a + K_a,$$
$$F_3 = \sigma(1 - I_aK_a),$$
$$\alpha_1 = \omega/k - \beta_1,$$
$$\alpha_2 = k\beta_1(kR + I_a) - I_a\omega - \beta_2,$$
$$\alpha_3 = k\big[\beta_2(kR + I_a) - I_a(kI_a\beta_1 + k^2R\beta_1 - I_a\omega)\big] - \beta_3,$$
$$\beta_1 = ikE_0^2\big[\varepsilon(\sigma-1)\gamma_1K_a + I_a\big(\delta_1(\sigma-1) - (\varepsilon-1)K_a\big) - 2i\mu I_aK_a\omega\big]/I_aK_a(2\mu k^2 - i\rho\omega),$$

$$\begin{aligned}\beta_2 = {}& -(2\mu k^2 - i\rho\omega)^{-2}\big\{2\mu k^3(s_1 - kR)\big(2k\mu\omega + ik\varepsilon\varepsilon_2E_0^2 \\ &\times \big[\big((I_a\delta_1 + K_a\varepsilon\gamma_1)(\sigma-1) + I_aK_a(\varepsilon-1)\big)\big]/I_aK_a\big)\big)\big\} \\ &+ (2\mu k^2 - i\rho\omega)^{-1}\big\{ik\varepsilon\varepsilon_2E_0^2(\sigma-1)\big(\gamma_2 - 2kI_a\gamma_1 \\ &+ k(\sigma-1)\gamma_1^2\big)/I_a + ik\varepsilon_2E_0^2(\sigma-1)\big(\delta_2 - 2kK_a\delta_1 \\ &+ k(\sigma-1)\delta_1^2\big)/K_a + 2k^2\mu s_1\omega\big\},\end{aligned}$$

$$\begin{aligned}\beta_3 = {}& -(2\mu k^2 - i\rho\omega)^{-2}\big\{2\mu k^3(s_1 - kR)\big[2k\mu\omega + ik\varepsilon\varepsilon_2E_0^2 \\ &\times (\sigma-1)\big(\gamma_2 + k\gamma_1\big(\gamma_1(\sigma-1) - 2I_a\big)\big)/I_a \\ &+ ik\varepsilon_2E_0^2(\sigma-1)\big(\delta_2 + k\delta_1\big(\delta_1(\sigma-1) - 2K_a\big)\big)/K_a\big]\big\} \\ &+ (2\mu k^2 - i\rho\omega)^{-1}\big\{\big[ik\varepsilon_2E_0^2(\sigma-1)\big(K_a\delta_3 + k\big(2K_a^2\delta_2^2 \\ &+ k\delta_1^2(1 + K_a^2)(\sigma-1) + K_a\delta_1\big(k + 2\delta_2(\sigma-1)\big)\big)\big)\big]/K_a^2 \\ &+ \big[ik\varepsilon\varepsilon_2E_0^2(\sigma-1)\big(I_a\gamma_3 + k\big(2I_a^2\gamma_2^2 + k\gamma_1^2\big(1 + I_a^2\big)(\sigma-1) \\ &+ I_a\gamma_1\big(k + 2\gamma_2(\sigma-1)\big)\big)\big)\big]/I_a^2 - 2k^3\mu(s_1I_a - 1)\omega\big\} \\ &- (2\mu k^2 - i\rho\omega)^{-2}\big\{\big[-i\varepsilon\varepsilon_2kE_0^2(1 - (\sigma-1)\gamma_1)/I_a \\ &+ i\varepsilon_2kE_0^2(1 - (\sigma-1)\delta_1)/K_a + 2k\mu\omega\big]\big[\big(2k^4 + 2k^4I_a^2 \\ &+ 2Rk^5I_a - 2k^4(s_1 + I_a)(kR + I_a)\big)\mu - i\rho\omega k^2\big]\big\}.\end{aligned}$$

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